MATH 1010E Week 1 (Martin Li)

Outline of the Course

Actual

Q: What is "Calculus" about?

- as a tool to understand "functions". $f(x) = x^2$, $\cos x$, $\log x$.

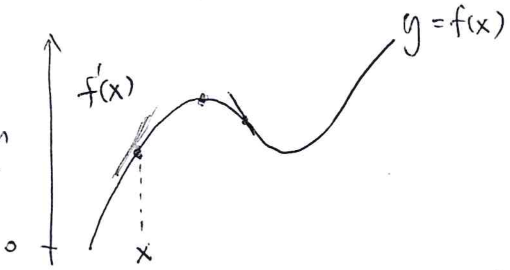
$\boxed{\text{Calculus}}$

Differentiation

$\frac{df}{dx}$, $f'(x)$

- rate of change, slope

mountain



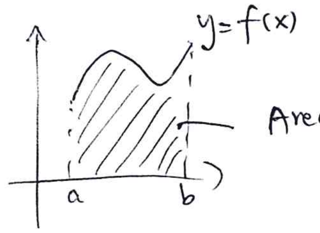
Q: how steep is the mountain?
- depends on where you stand.
- $f'(x)$ = slope at x.

- Newton (mechanics)
- position $\xrightarrow[\text{differentiate}]{}$ velocity

Integration

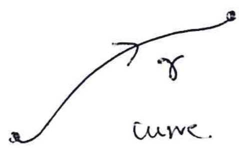
$\int f(x)\,dx$, $\int_a^b f(x)\,dx$

- area



Area $= \int_a^b f(x)\,dx$.

- length of a curve.



Differentiation $\xleftrightarrow[\text{Theorem of Calculus}]{\text{Fundamental}}$ Integration

Plan of this Class

1) What are "functions"? Elementary functions?

2) What is "$\lim_{x \to a} f(x)$"? Concept of "∞".

3) What is differentiation "$\frac{df}{dx}$" or "$f'(x)$"?

4) What is integration "$\int f\,dx$"?

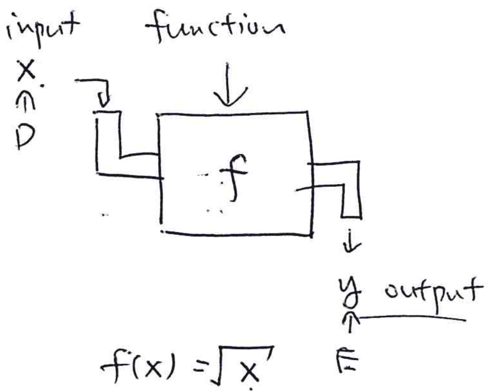
5) Fundamental Thm. of Calculus: "$\int \frac{df}{dx}\,dx = f(x)$"

or "$\frac{d}{dx}\left(\int_a^x f\,dx\right) = f(x)$"

Chapter 1 Functions

E.g. $f(x) = x$, $f(x) = 2x+1$, $f(x) = ax^2 + bx + c$,

$f(x) = \cos x$, $f(x) = \log_{10} x$, ...



Q: What is a function?

"function" f
- domain D (input)
- codomain E (output)
- a rule of assigning every $x \in D$ a unique output $y \in E$.

$f(x) = \sqrt{x}$

"$\in$" = "in"

belongs

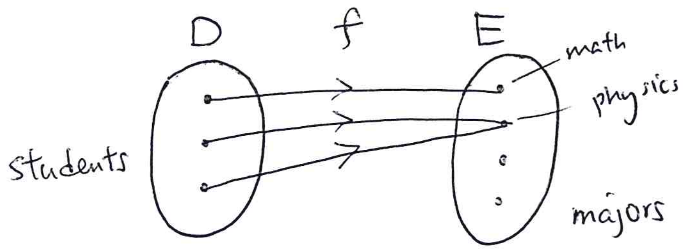
Examples: (1) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x$.

$\mathbb{R} = D$, $\mathbb{R} = E$

(2) $f : [0, \infty) \to \mathbb{R}$, $f(x) = x$

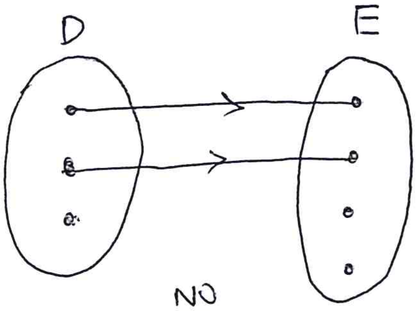
$[0,\infty) = D$, $\mathbb{R} = E$

Diagram:



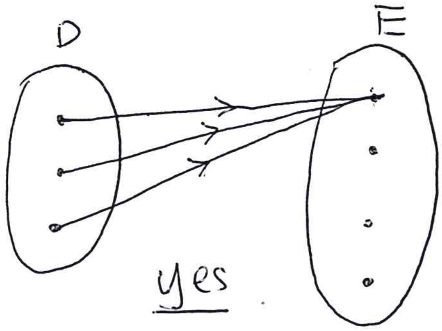
Q: Which of the following are functions ?

(1)



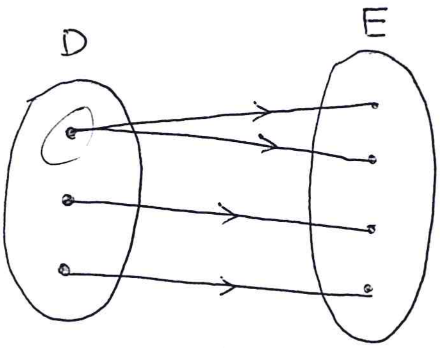
NO

(2)



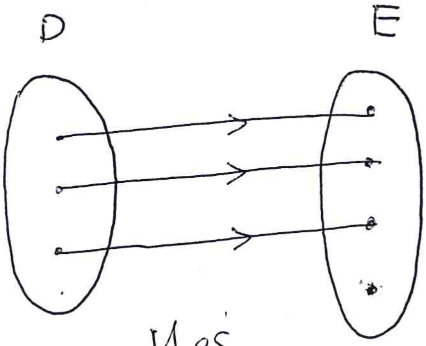
yes.

(3)



NO.

(4)



yes

Some notations (for lazy mathematicians).

$\mathbb{R}$: the set of all real numbers e.g. $0, 1, 1.5, \pi, -1, \dots$

$\mathbb{N}$: the set of all natural numbers e.g. $1, 2, 3, 4, 5, \dots$

$\in$: "belongs to" e.g. $1 \in \mathbb{R}$, $2 \in \mathbb{N}$.

$[a,b]$: closed interval $\{ x \in \mathbb{R} \mid a \leq x \leq b \}$, e.g. $[0,1] \ni \frac{1}{2}, 1$

(a,b) : open interval $\{ x \in \mathbb{R} \mid a < x < b \}$, e.g. $(0,1) \ni \frac{1}{2}$, $\not\ni 1$

$[a,b)$, $(a,b]$: half intervals

$\exists$: "there exists" / "for some"

$\forall$: "for any" / "for all"

$\exists!$: "there exists a unique"

$:=$: "defined as" e.g. $f(x) := x^2$

$\Rightarrow$: "implies", "if ..., then ...." e.g. $A \Rightarrow B$.

$\Leftrightarrow$: "equivalent to", "if and only if"

s.t. : "such that" / "so that"

Example : "For any real no. x greater than 0, there is a unique real no. y greater than 0 such that the square of y is equal to x".

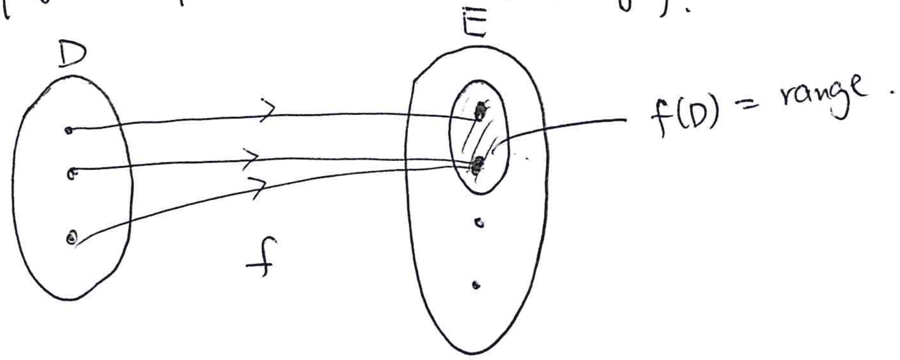
OR "$\forall x \in \mathbb{R}, x \geq 0, \exists! y \in \mathbb{R}, y \geq 0$ s.t. $y^2 = x$".

Ex: Write down the definition of a "function" using these notations.

Range and Codomain

Given $f: D \to E$ a function, the "range" of f to be

$$f(D) := \{ y \in E \mid \exists\, x \in D \text{ s.t. } f(x) = y \}.$$

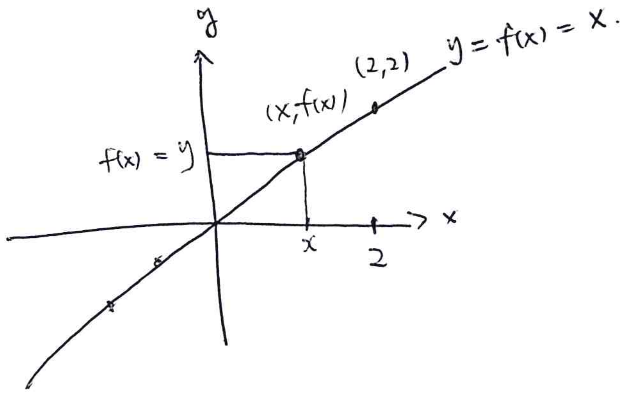


Ex: What is the range of $f: \mathbb{R} \to \mathbb{R}$ $f(x) = x^2$?

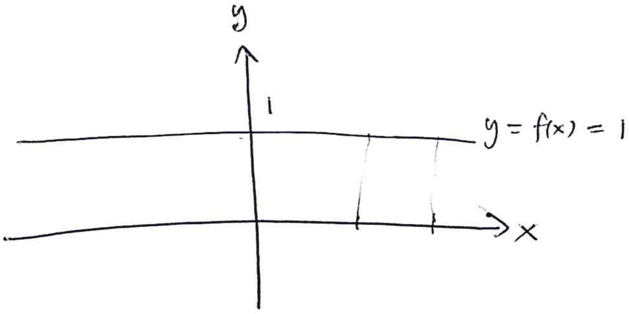
Ans: range $= [0, \infty)$.

Functions as "formula" / "graph"

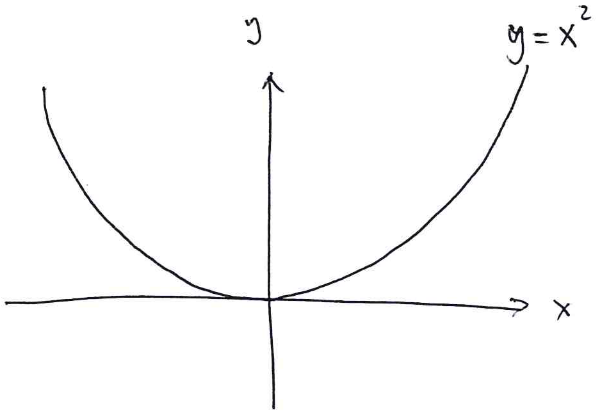
(1) $f: \mathbb{R} \to \mathbb{R}$ $f(x) = x$.



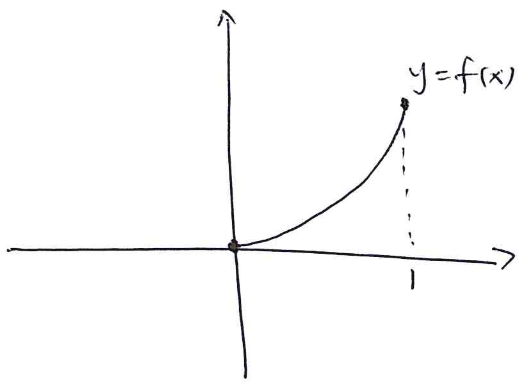
(2) $f: \mathbb{R} \to \mathbb{R}$ $f(x) = 1$



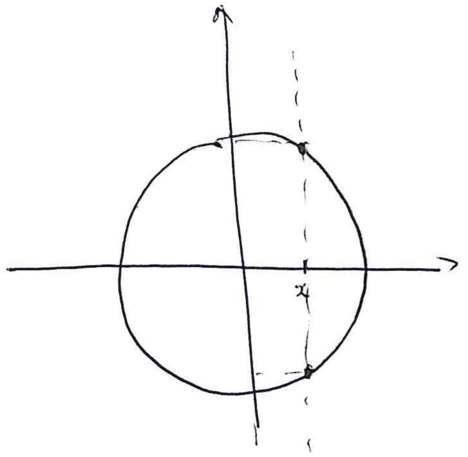
(3) $f: \mathbb{R} \to \mathbb{R}$ $f(x) = x^2$



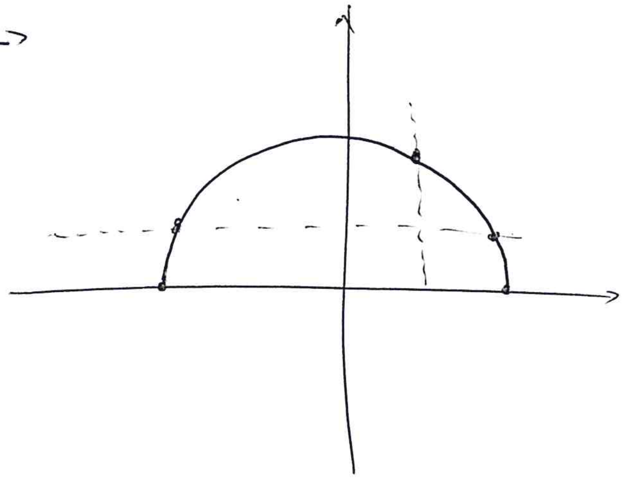
(4) $f: [0,1] \to \mathbb{R}$ $f(x) = x^2$

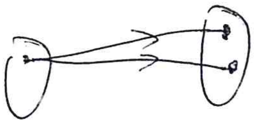


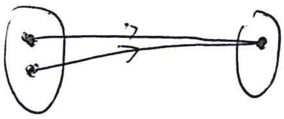
Ex: what is the graph of $f(x) = x^3$, $x^2 + x - 1$, $\frac{1}{x}$, $x^3 + x^2 - x - 1$, $\log x$?

Q: Is  the graph of a function?

Ans: no! not "uniqueness"!



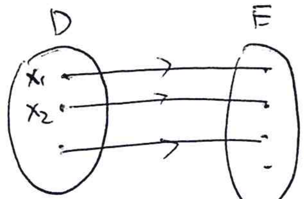




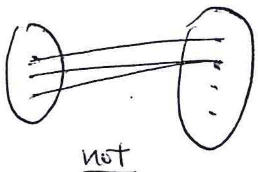
Injective / Surjective Functions

Def: A function $f: D \to E$ is

(i) injective / 1-1 if no two elements in D are assigned to the same element in E.



1-1



not 1-1.

more precisely.

" $\forall\, x_1, x_2 \in D$, $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$ in E "

or (contrapositive)

" $f(x_1) = f(x_2)$ $x_1, x_2 \in D \Rightarrow x_1 = x_2$ in D "

↑ use to prove a function is 1-1.

$A \Rightarrow B$

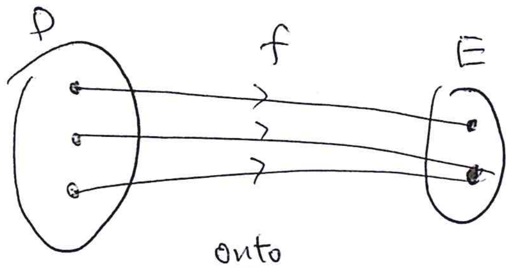
$\Updownarrow$

not B $\Rightarrow$ not A.

Ex: $\not\Updownarrow$

$B \Rightarrow A$.

(ii) surjective / onto if no elements in E is being left over.



onto

or

" $\forall y \in E, \exists x \in D$ st. $f(x) = y$ ".

eg. If $f(x) = x^2 + 2x - 1$, then given $1 \in \mathbb{R}$,

$\exists x \in \mathbb{R}$ st $f(x) = 1$?

Example: Show that $f: [0, \infty) \to \mathbb{R}$, $f(x) = x^2$

is 1-1 but not onto. (Ex: What if I change the domain / codomain?)

(i) f is 1-1 : $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$

$f(x_1) = f(x_2) \implies x_1^2 = x_2^2$ (by def. of f)

$\implies x_1 = x_2$ (by taking $\sqrt{\ }$ on both sides, and $x_1, x_2 \in D$, $x_1, x_2 \geq 0$)

1-1.

(ii) f is not onto : take $y = -1$.

then $\nexists x \in [0, \infty)$ st $f(x) = x^2 = -1$ not onto.

Q: from students

1) onto $\iff$ range = codomain

2) notation: $f(x)$ as function or image.

Last time:
- function $f: D \longrightarrow E$, domain, codomain, range.
  graph of f.
- injective (1-1) / surjective (onto).
- error: "greater than" $\Leftrightarrow$ "$>$".
  "greater than or equal to" $\Leftrightarrow$ "$\geq$"

One More Example Consider the function

$$f: D \longrightarrow \mathbb{R} \quad \text{codomain} \qquad D \subset \mathbb{R}.$$

$$\boxed{f(x) = \frac{9x+1}{x^2-3x+2}}$$

(1) Find the maximum domain of definition. (i.e. largest D).
and the range with this max. domain.

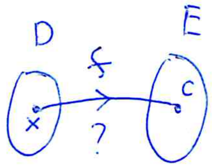
(2) Is f 1-1 or onto? Prove it.

Solution: (1) The function makes sense except when denominator $= 0$.

$$\Leftrightarrow \quad x^2-3x+2=0.$$

$$\Leftrightarrow \quad (x-1)(x-2)=0$$

$$\Leftrightarrow \quad x = 1 \text{ or } 2$$



The max domain $D = \mathbb{R} \setminus \{1,2\}$. *
without

If $c \in \text{range}(f)$. (def: $\exists\, x \in D$ s.t. $f(x) = c$).

$$f(x) = c \quad \Leftrightarrow \quad \frac{9x+1}{x^2-3x+2} = c$$

$$\Leftrightarrow \quad \underbrace{cx^2 + (-3c-9)x + (2c-1) = 0}_{\text{quadratic eq}^{\underline{n}}}.$$

quadratic eq$^{\underline{n}}$: c given, solve for x?

$\Leftrightarrow$ discriminant $= \Delta \geq 0$.

$$(-3c-\overset{a}{\cancel{9}})^2 - 4c(2c-\overset{b}{\cancel{1}}) \geq 0.$$

$$\Leftrightarrow \quad 9c^2 + \overset{6a}{\cancel{54}}c + \overset{a^2}{\cancel{81}} - 8c^2 + \overset{4b}{\cancel{4c}} \geq 0.$$

$$\Leftrightarrow \quad c^2 + \overset{6a+4b}{\cancel{58}}c + \overset{a^2}{\cancel{81}} \geq 0.$$

$a=5 \quad -16.$

$30\cancel{4} \; -22 \quad b = -\frac{22}{4} = -\frac{11}{2}$

$$\Leftrightarrow \quad (c+29)^2 - 29^2 + 81 \geq 0.$$

$10x-11$

$$\Leftrightarrow \quad c \geq \sqrt{(29^2-81)} - 29$$

$\Rightarrow$ range (f) found.

$$\text{or} \quad c \leq -\sqrt{(29^2-81)} - 29.$$

(2). f onto ? (Recall: f onto $\Leftrightarrow$ range(f) $=$ codomain(f))

<u>NO</u>! range(f) $\neq \mathbb{R}$, codomain(f) $= \mathbb{R}$

f 1-1 ? for any $c \neq \pm\sqrt{(29^2-81)} - 29$.

<u>NO</u>!

$(3c+5)^2 - 2c(4c+11)$

$c^2+8c+25 \geq 0$

$c^2+8c+16+9$

<u>Elementary Functions</u>

(A) <u>Polynomials</u> : $f: \mathbb{R} \to \mathbb{R}$.

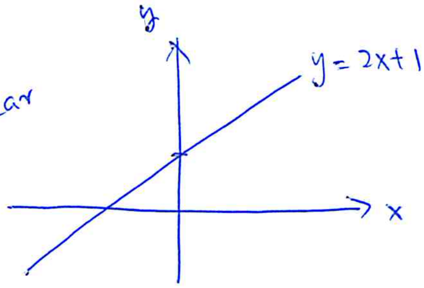
$$f(x) = \sum_{k=0}^{n} a_k x^k = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n.$$

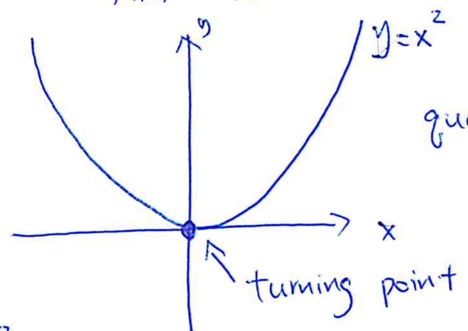
given real numbers.

degree n polynomial when $a_n \neq 0$.

<u>Eg.</u> $f(x) = 2x+1$ $\qquad f(x) = x^2$

linear





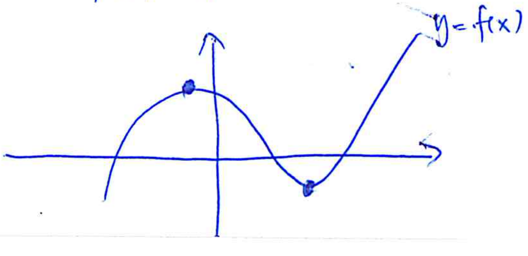
quadratic.

$(x+2)(x-1)(x-2)$

$(x^2-4)(x-1)$

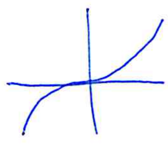
$f(x) = x^3 + 2x^2 - x - 1$

cubic



<u>Q</u>: Prove that a "general" degree n polynomial has (n-1) "turning points".

<u>eg.</u> $f(x) = x^3$



(B) <u>Exponential / Logarithm</u>

$$\exp : \mathbb{R} \to \mathbb{R}$$

$$(e^x =)\exp(x) := \sum_{k=0}^{\infty} \frac{x^k}{k!} = \underbrace{1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots}_{\text{infinite series!}} \overset{x=1}{=:} e$$

$$e := \overset{f(1)}{\exp(1)} \overset{?}{=} \boxed{\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n}$$

<u>Fact</u>: The infinite series above makes sense for any $x \in \mathbb{R}$.

infinite series can be bad : Eg. $1-1+1-1+1-1 \cdots\cdots = 1$ or 0 ?

<u>none!</u>

<u>Recall</u>: $10^x = y$ $\Updownarrow$ $x = \log_{10} y$, $y > 0$.

looks different!

$\ln_e$ $\log_{10}$ $\log$

$$\ln : \mathbb{R}_{>0} \to \mathbb{R}.$$

$$\ln(\underbrace{1+x}_{>0}) \boxed{:=} \sum_{k=1}^{\infty} (-1)^{k-1} \frac{x^k}{k}$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots\cdots$$ holds for <u>$x > -1$</u>

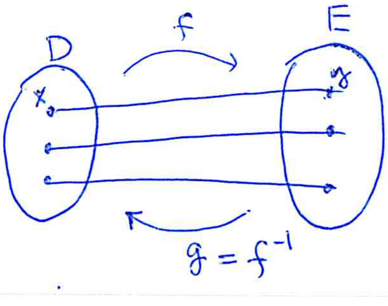
$-2 - \frac{4}{2} - \frac{8}{3} = -\infty.$ <u>$x = -2$</u>

<u>Fact</u>: (1) $\exp(\ln y) = y \quad \forall y > 0$

(2) $\ln(\exp(x)) = x \quad \forall x \in \mathbb{R}.$

<u>Def</u>: Two functions $f: D \to E$ & $g: E \to D$ are <u>inverse</u> to each other if (i) $g(f(x)) = x \quad \forall x \in D$

(ii) $f(g(y)) = y \quad \forall y \in E$



<u>Q</u>: If f & g are inverse, then f is 1-1 & onto.

if f is 1-1 & onto then inverse $g = f^{-1}$ exists.

Properties of $\exp / e^x$:

(A) $\boxed{e^{x+y} = e^x \cdot e^y}$

"Pf". R.H.S. $= e^x \cdot e^y$

$$= \left(1+x+\frac{x^2}{2}+\cdots\cdots\right)\cdot\left(1+y+\frac{y^2}{2}+\cdots\cdots\right)$$

$$\text{"="}\ 1 + x + \frac{x^2}{2} + \cdots$$

$$+y + xy + \frac{x^2y}{2} + \cdots$$

$$+\frac{y^2}{2} + \frac{xy^2}{2} + \frac{x^2y^2}{2} + \cdots$$

$$= 1 + (x+y) + \frac{1}{2}(x+y)^2 + \cdots\cdots = e^{(x+y)}$$

Ex: Prove this more formally using binomial formula.